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THE CURVATURE PROPERTIES IN A FIVE-DIMENSIONAL FINSLER SPACE IN TERMS OF SCALARS

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ABSTRACT

The present paper refers to the study of the properties of torsion tensors C_{ijk} and P_{ijk} and also the third curvature tensor in a Five dimensional Finsler space based on theory of orthonormal frame.

KEYWORDS: Curvature & Five-Dimensional Spaces

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Original Article

INTRODUCTION

Based on the intrinsic field of an orthonormal frame consisting of the normalized support element Li and the unit vector m^i, normal to lie, Berwald [1, 2], developed the study of two dimensional Finsler spaces. Following the Berwalds idea of two-dimensional case Moor [8], introduced in a three- dimensional Finsler space the intrinsic field of orthonormal frame, which consists of l^i, the normalized torsion vector m^i and the unit vector n^i, orthogonal to both of them. Three-dimensional Finsler spaces and their various aspects have been studied by several authors, namely Nobuchara and Nagai [9], Matsumoto [4, 5,6], Rastogi [12, 13], Rastogi and Dwivedi [15] and many others. Similar to three-dimensional Finsler spaces, four-dimensional Finsler spaces have been developed and studied by Rastogi [16], Pandey and Dwivedi [10] and others. Pandey, Dwivedi and Gupta [11] initiated study of five dimensional Finsler spaces in terms of scalars.

The purpose of the present paper is to study the properties of torsion tensors C_{ijk} and $P_{ijk}$and also the third curvature tensor in a five-dimensional Finsler space based on the theory of orthonormal frame fields.

PRELIMINARIES

Let F^5 be a five-dimensional Finsler space equipped with a fundamental function L(x,y), orthonormal frame $e_{\alpha)I}$, (α = 1,2,3,4,5), adopted components of metric tensor g_{ij} and E-tensors ε_{ijklm} respectively given by $\delta_{\alpha\beta}$ and $\varepsilon_{\alpha\beta\gamma\delta\Theta} = (\delta^{1}{}_{\alpha}{}^{2}{}_{\beta}{}^{3}{}_{\gamma}{}^{4}{}_{\delta}{}^{5}{}_{\Theta})$, where generalised Kronecker delta satisfies

$$\gamma_{ijklm} = \delta^{1\,2\,3\,4\,5}_{i\,j\,k\,l\,m},\ \gamma^{ijklm} = \delta^{i\,j\,k\,l\,m}_{1\,2\,3\,4\,5} \tag{2.1}$$

then the E-tensors are defined by

$$\varepsilon_{ijklm} = |g|^{1/2}\gamma_{ijklm},\ \varepsilon^{ijklm} = |g|^{-1/2}\gamma^{ijklm}, \tag{2.2}$$

$|g|$ being the determinant consisting of the components of the fundamental tensor $g = (g_{ij})$.

In a five -dimensional Finsler space F^5, we have five ortho normal unit vectors, which shall be denoted by l_i, m_i, n_i, p_i and q_i. The h- covariant derivative $e_{\alpha)}{}^{i}{}_{/j}$ of the vector$e_{\alpha)}{}^{i}$, belonging to Miron frame can be given as

$$e_{1)}{}^{i}{}_{/j} = l^{i}{}_{/j} = 0,\ e_{2)}{}^{i}{}_{/j} = m^{i}{}_{/j} = n^{i}h_j - p^{i}\, u_j - q^{i}\, r_j,\ e_{3)}{}^{i}{}_{/j} = n^{i}{}_{/j} = p^{i}\, k_j - m^{i}\, h_j - q^{i}\, s_j,$$

$$e_{4)}{}^{i}{}_{/j} = p^{i}{}_{/j} = m^{i}\, u_j - n^{i}\, k_j - q^{i}\, t_j,\ e_{5)}{}^{i}{}_{/j} = q^{i}{}_{/j} = m^{i}\, r_j + n^{i}\, s_j + p^{i}\, t_j \tag{2.3}$$

where h_j, k_j, u_j, r_j, s_j and t_j are called h-connection vectors of F^5.

Also the v-covariant derivative $e_{\alpha)}{}^{i}{}_{//j}$ of the vector $e_{\alpha)}{}^{I}$ belonging to Miron frame $e_{\alpha)}$, can be given as

$$L\, e_{1)}{}^{i}{}_{//j} = L\, l^{i}{}_{//j} = h^{i}{}_{j} = m^{i}\, m_j + n^{i}\, n_j + p^{i}\, p_j + q^{i}\, q_j,$$

$$L\, e_{2)}{}^{i}{}_{//j} = L\, m^{i}{}_{//j} = - l^{i}\, m_j + n^{i}\, U_j + p^{i}\, V_j + q^{i}\, X_j,$$

$$L\, e_{3)}{}^{i}{}_{//j} = L\, n^{i}{}_{//j} = - l^{i}\, n_j - m^{i}\, U_j + p^{i}\, W_j + q^{i}\, Y_j,$$

$$L\, e_{4)}{}^{i}{}_{//j} = L\, p^{i}{}_{//j} = - l^{i}\, p_j - m^{i}\, V_j - n^{i}\, W_j + q^{i}\, Z_j,$$

$$L\, e_{5)}{}^{i}{}_{//j} = L\, q^{i}{}_{//j} = - l^{i}\, q_j - m^{i}\, X_j - n^{i}\, Y_j - p^{i}\, Z_j \tag{2.4}$$

where U_j, V_j, W_j, X_j, Y_j and Z_j are called v-connection vectors.

Any third order symmetric tensor C_{ijk} satisfying $C_{ijk}\, l^i = 0$, in a five- dimensional Finsler space F^5, can be expressed as

$$L\, C_{ijk} = C_{(1)}\, m_i\, m_j\, m_k + C_{(2)}\, n_i\, n_j\, n_k + C_{(3)}\, p_i\, p_j\, p_k + C_{(4)}\, q_i\, q_j\, q_k$$

$$+ \sum_{(I,j,k)} [\, C_{(5)}\, m_i\, m_j\, n_k + C_{(6)}\, m_i\, m_j\, p_k + C_{(7)}\, m_i\, m_j\, q_k + C_{(8)}\, n_i\, n_j\, m_k$$

$$+ C_{(9)}\, n_i\, n_j\, p_k + C_{(10)}\, n_i\, n_j\, q_k + C_{(11)}\, p_i\, p_j\, m_k + C_{(12)}\, p_i\, p_j\, n_k$$

$$+ C_{(13)}\, p_i\, p_j\, q_k + C_{(14)}\, q_i\, q_j\, m_k + C_{(15)}\, q_i\, q_j\, n_k + C_{(16)}\, q_i\, q_j\, p_k$$

$$+ C_{(17)}\, m_i\, (n_j\, p_k + n_k\, p_j) + C_{(18)}\, m_i\, (n_j\, q_k + n_k\, q_j)$$

$$+ C_{(19)}\, m_i\, (p_j\, q_k + p_k\, q_j) + C_{(20)}\, n_i\, (p_j\, q_k + p_k\, q_j) \tag{2.5}$$

where

$$C_{(1)} + C_{(8)} + C_{(11)} + C_{(14)} = L\, C,\ C_{(2)} + C_{(5)} + C_{(12)} + C_{(15)} = 0,$$

$$C_{(3)} + C_{(6)} + C_{(9)} + C_{(16)} = 0,\ C_{(4)} + C_{(7)} + C_{(10)} + C_{(13)} = 0 \tag{2.6}$$

and $C_{(17)}$, $C_{(18)}$, $C_{(19)}$ and $C_{(20)}$ are non-zero scalars in F^5.

Alternatively [11], since $C_{\alpha\beta\gamma} = L\, C_{ijk} e_{\alpha)}{}^{i} e_{\beta)}{}^{j}\, e_{\gamma)}{}^{k}$, we have

$$C_{1\beta\gamma} = 0,\ C_{2\beta\beta} = LC,\ C_{3\beta\beta} = 0,\ C_{4\beta\beta} = 0,\ C_{5\beta\beta} = 0,$$

$$C_{222} + C_{233} + C_{244} + C_{255} = LC,\ C_{322} + C_{333} + C_{344} + C_{355} = 0,$$

$$C_{422} + C_{433} + C_{444} + C_{455} = 0,\ C_{522} + C_{533} + C_{544} + C_{555} = 0. \tag{2.7}$$

Also $C_{234} \neq 0$, $C_{235} \neq 0$, $C_{245} \neq 0$, $C_{345} \neq 0$ and

$C_{(17)} = C_{234} = L\, C_{ijk}\, m^i n^j\, p^k,\ C_{(18)} = C_{235} = L\, C_{ijk}\, m^i\, n^j\, q^k,$

$C_{(19)} = C_{245} = L\, C_{ijk}\, m^i\, p^j\, q^k,\ C_{(20)} = C_{345} = L\, C_{ijk}\, n^i p^j\, q^k$ (2.8)

Torsion Tensor P_{ijk} in F^5

Using the definition $P_{ijk} = LC_{ijk/0}$, after a very tedious calculation on rearranging terms we can obtain

$$
\begin{aligned}
P_{ijk} = {} & L[C_{(1)/0}\, m_i\, m_j\, m_k + C_{(2)/0}\, n_i\, n_j\, n_k + C_{(3)/0}\, p_i\, p_j\, p_k + C_{(4)/0}\, q_i\, q_j\, q_k \\
& + \textstyle\sum_{(i,j,k)} [m_i\, m_j\, \{C_{(1)}\, m_{k/0} + (C_{(5)}\, n_k)_{/0} + (C_{(6)}\, p_k)_{/0} + (C_{(7)}\, q_k)_{/0}\} \\
& + n_i\, n_j\, \{C_{(2)}\, n_{k/0} + (C_{(8)} m_k)_{/0} + (C_{(9)}\, p_k)_{/0} + (C_{(10)} q_k)_{/0}\} \\
& + p_i\, p_j\, \{C_{(3)}\, p_{k/0} + (C_{(11)}\, m_k)_{/0} + (C_{(12)}\, n_k)_{/0} + (C_{(13)}\, q_k)_{/0}\} \\
& + q_i\, q_j\, \{C_{(4)}\, q_{k/0} + (C_{(14)}\, m_k)_{/0} + (C_{(15)}\, n_k)_{/0} + (C_{(16)}\, p_k)_{/0}\} \\
& + (m_i\, n_j + m_j\, n_i)\{C_{(5)}\, m_{k/0} + C_{(8)}\, n_{k/0} + C_{(17)}\, p_{k/0} + (C_{(18)}\, q_k)_{/0}\} \\
& + (m_i\, p_j + m_j\, p_i)\{C_{(6)}\, m_{k/0} + C_{(11)}\, p_{k/0} + C_{(17)}\, n_{k/0} + C_{(19)}\, q_{k/0}\} \\
& + (m_i\, q_j + m_j\, q_i)\{C_{(7)}\, m_{k/0} + C_{(14)}\, q_{k/0} + C_{(18)}\, n_{k/0} + C_{(19)}\, p_{k/0}\} \\
& + (n_i\, p_j + n_j\, p_i)\{C_{(9)}\, n_{k/0} + C_{(12)}\, p_{k/0} + (C_{(17)}\, m_k)_{/0} + C_{(20)}\, q_{k/0}\} \\
& + (n_i\, q_j + n_j\, q_i)\{C_{(10)}\, n_{k/0} + C_{(15)}\, q_{k/0} + C_{(18)}\, m_{k/0} + C_{(20)}\, p_{k/0}\} \\
& + (p_i\, q_j + p_j\, q_i)\{C_{(13)}\, p_{k/0} + C_{(16)}\, q_{k/0} + (C_{(19)}\, m_k)_{/0} + (C_{(20)}\, n_k)_{/0}\}]] \qquad (3.1)
\end{aligned}
$$

Since we know that, we can write

$$
\begin{aligned}
& m_{k/0} = n_k\, h_0 - p_k\, u_0 - q_k\, r_0,\ n_{k/0} = p_k\, k_0 - m_k\, h_0 - q_k\, s_0, \\
& p_{k/0} = m_k\, u_0 - n_k\, k_0 - q_k\, t_0,\ q_{k/0} = m_k\, r_0 + n_k\, s_0 + p_k\, t_0 \qquad (3.2)
\end{aligned}
$$

therefore, on substituting these values in (3.1), we get on simplification

$$
\begin{aligned}
P_{ijk} = {} & L[\{C_{(1)/0} + 3(C_{(6)}\, u_0 - C_{(5)}\, h_0 + C_{(7)}\, r_0)\}m_i\, m_j\, m_k + \{C_{(2)/0} + 3(C_{(8)}\, h_0 - C_{(9)}\, k_0 \\
& + C_{(10)}\, s_0)\}n_i\, n_j\, n_k + \{C_{(3)/0} + 3(C_{(12)}\, k_0 - C_{(11)}\, u_0 + C_{(13)}\, t_0)\}p_i\, p_j\, p_k \\
& + \{C_{(4)/0} - 3(C_{(14)}\, r_0 + C_{(15)}\, s_0 + C_{(16)}\, t_0)\}q_i\, q_j\, q_k \\
& + \textstyle\sum_{(i,j,k)}[m_i\, m_j\, n_k\, \{C_{(5)/0} + (C_{(1)} - 2C_{(8)})h_0 - C_{(6)}\, k_0 + C_{(7)}\, s_0 \\
& + 2\, C_{(17)}\, u_0 + 2C_{(18)}\, r_0\} + m_i\, m_j\, p_k\{C_{(6)/0} - (C_{(1)} - 2\, C_{(11)})u_0 \\
& + C_{(5)}\, k_0 + C_{(7)}\, t_0 - 2\, C_{(17)}\, h_0 + 2\, C_{(19)}\, r_0\} + m_i\, m_j\, q_k\, \{C_{(7)/0} \\
& -(C_{(1)} - 2\, C_{(14)})r_0 - C_{(5)}\, s_0 - C_{(6)}\, t_0 - 2\, C_{(18)}\, h_0 + 2\, C_{(19)}\, u_0\} \\
& + n_i\, n_j\, m_k\{C_{(8)/0} - (C_{(2)} - 2\, C_{(5)})h_0 + C_{(9)}\, u_0 + C_{(10)}\, r_0 - 2\, C_{(17)}\, k_0 \\
& + 2\, C_{(18)}\, s_0\} + n_i\, n_j\, p_k\, \{C_{(9)/0} + (C_{(2)} - 2C_{(12)}\, k_0 - C_{(8)}\, u_0 + C_{(10)}\, t_0 \\
& + 2\, C_{(17)} h_0 + 2\, C_{(20)}\, s_0\} + n_i\, n_j\, q_k\, \{C_{(10)/0} - (C_{(2)} - 2\, C_{(15)})s_0 - C_{(8)}\, r_0
\end{aligned}
$$

$-C_{(9)} t_0 + 2 C_{(18)} h_0 - 2 C_{(20)} k_0\} + p_i p_j m_k \{C_{(11)/0} + (C_{(3)} - 2 C_{(6)})u_0$

$- C_{(12)} h_0 + C_{(13)} r_0 + 2C_{(17)} k_0 + 2C_{(19)} t_0\} + p_i p_j n_k \{C_{(12)/0} + C_{(11)} h_0$

$-(C_{(3)} - 2 C_{(9)})k_0 + C_{(13)} s_0 - 2 C_{(17)} u_0 + 2 C_{(20)} t_0\} + p_i p_j q_k \{C_{(13)/0}$

$-(C_{(3)} - 2 C_{(16)}) t_0 - C_{(11)} r_0 - C_{(12)} s_0 - 2 C_{(19)} u_0 + 2 C_{(20)} k_0\}$

$+ q_i q_j m_k \{C_{(14)/0} + (C_{(4)} - 2 C_{(7)})r_0 - C_{(15)} h_0 + C_{(16)} u_0 - 2 C_{(18)} s_0$

$-2 C_{(19)} t_0\} + q_i q_j n_k \{C_{(15)/0} + (C_{(4)} - 2 C_{(10)}) s_0 + C_{(14)} h_0 - C_{(16)} k_0$

$-2 C_{(18)} r_0 - 2 C_{(20)} t_0\} + q_i q_j p_k \{C_{(16)/0} + (C_{(4)} - 2 C_{(13)})t_0 - C_{(14)} u_0$

$+ C_{(15)} k_0 - 2 C_{(19)} r_0 - 2 C_{(20)} s_0\} + (m_i n_j + m_j n_i) p_k\{C_{(17)/0} - C_{(5)} u_0$

$+ (C_{(8)} - C_{(11)})k_0 + (C_{(6)} - C_{(9)})h_0 + C_{(12)} u_0 + C_{(18)} t_0 + C_{(19)} s_0 + C_{(20)} r_0\}$

$+ (m_i n_j + m_j n_i)q_k \{C_{(18)/0} - (C_{(5)} - C_{(15)})r_0 - (C_{(8)} - C_{(14)}) s_0 - C_{(17)} t_0$

$+ (C_{(7)} - C_{(10)})h_0 - C_{(19)} k_0 + C_{(20)} u_0\} + (m_i p_j + m_j p_i)q_k \{C_{(19)/0} - C_{(17)} s_0$

$-(C_{(7)} - C_{(13)})u_0 - (C_{(6)} - C_{(16)})r_0 - (C_{(11)} - C_{(14)})t_0 + C_{(18)} k_0 - C_{(20)} h_0\}$

$+ (n_i p_j + n_j p_i) q_k \{C_{(20)/0} + (C_{(10)} - C_{(13)}) k_0 - (C_{(9)} - C_{(16)})s_0 - C_{(17)} r_0$

$-(C_{(12)}- C_{(15)}) t_0 - C_{(18)} u_0 + C_{(19)} h_0\}]]$ (3.3)

It is known Izumi [3], that in a P*-Finsler space for a constant λ, the tensor P_{ijk} is related to C_{ijk} as follows:

$P_{ijk} = \lambda C_{ijk},$ (3.4)

therefore, on substituting the values of P_{ijk} and C_{ijk} we get

$C_{(1)/0} + 3(C_{(6)} u_0 - C_{(5)} h_0 + C_{(7)} r_0) = \lambda C_{(1)}$

$C_{(2)/0} + 3(C_{(8)} h_0 - C_{(9)} k_0 + C_{(10)} s_0) = \lambda C_{(2)}$ $\quad C_{(3)/0} + 3(C_{(12)} k_0 - C_{(11)} u_0 + C_{(13)} t_0) = \lambda C_{(3)}$

$C_{(4)/0} - 3(C_{(14)} r_0 + C_{(15)} s_0 + C_{(16)} t_0) = \lambda C_{(4)}$

$C_{(5)/0} + (C_{(1)} - 2C_{(8)})h_0 - C_{(6)} k_0 + C_{(7)} s_0 + 2 C_{(17)} u_0 + 2C_{(18)} r_0 = \lambda C_{(5)}$

$C_{(6)/0} - (C_{(1)} - 2 C_{(11)})u_0 + C_{(5)} k_0 + C_{(7)} t_0 - 2 C_{(17)} h_0 + 2 C_{(19)} r = \lambda C_{(6)}$

$C_{(7)/0}-(C_{(1)} - 2 C_{(14)})r_0 - C_{(5)} s_0 - C_{(6)} t_0 - 2 C_{(18)} h_0 + 2 C_{(19)} u_0 = \lambda C_{(7)}$

$C_{(8)/0} - (C_{(2)} - 2 C_{(5)})h_0 + C_{(9)} u_0 + C_{(10)} r_0 - 2 C_{(17)} k_0 + 2 C_{(18)} s_0 = \lambda C_{(8)}$

$C_{(9)/0} + (C_{(2)} - 2C_{(12)} k_0 - C_{(8)} u_0 + C_{(10)} t_0 + 2 C_{(17)} h_0 + 2 C_{(20)} s_0 = \lambda C_{(9)}$

$C_{(10)/0} - (C_{(2)} - 2 C_{(15)})s_0 - C_{(8)} r_0 -C_{(9)} t_0 + 2 C_{(18)} h_0 - 2 C_{(20)} k_0 = \lambda C_{(10)}$

$C_{(11)/0} + (C_{(3)} - 2 C_{(6)})u_0 - C_{(12)} h_0 + C_{(13)} r_0 + 2C_{(17)} k_0 + 2C_{(19)} t_0 = \lambda C_{(11)}$

$C_{(12)/0} + C_{(11)} h_0-(C_{(3)} - 2 C_{(9)})k_0 + C_{(13)} s_0 - 2 C_{(17)} u_0 + 2 C_{(20)} t_0\} = \lambda C_{(12)}$

$C_{(13)/0} -(C_{(3)} - 2 C_{(16)}) t_0 - C_{(11)} r_0 - C_{(12)} s_0 - 2 C_{(19)} u_0 + 2 C_{(20)} k_0 = \lambda C_{(13)}$

$C_{(14)/0} + (C_{(4)} - 2\, C_{(7)})r_0 - C_{(15)}\, h_0 + C_{(16)}\, u_0 - 2\, C_{(18)}\, s_0 - 2\, C_{(19)}\, t_0 = \lambda\, C_{(14)}$

$C_{(15)/0} + (C_{(4)} - 2\, C_{(10)})\, s_0 + C_{(14)}\, h_0 - C_{(16)}\, k_0 - 2\, C_{(18)}\, r_0 - 2\, C_{(20)}\, t_0 = \lambda\, C_{(15)}$

$C_{(16)/0} + (C_{(4)} - 2\, C_{(13)})t_0 - C_{(14)}\, u_0 + C_{(15)}\, k_0 - 2\, C_{(19)}\, r_0 - 2\, C_{(20)}\, s_0 = \lambda\, C_{(16)}$

$\{C_{(17)/0} - C_{(5)}\, u_0 + (C_{(8)} - C_{(11)})k_0 + (C_{(6)} - C_{(9)})h_0$

$+ C_{(12)}\, u_0 + C_{(18)}\, t_0 + C_{(19)}\, s_0 + C_{(20)}\, r_0\} = \lambda\, C_{(17)}$

$\{C_{(18)/0} - (C_{(5)} - C_{(15)})r_0 - (C_{(8)} - C_{(14)})\, s_0$

$C_{(17)}\, t_0 + (C_{(7)} - C_{(10)})h_0 - C_{(19)}\, k_0 + C_{(20)}\, u_0\} = \lambda\, C_{(18)}$

$\{C_{(19)/0} - C_{(17)}\, s_0 - (C_{(7)} - C_{(13)})u_0$

$- (C_{(6)} - C_{(16)})r_0 - (C_{(11)} - C_{(14)})t_0 + C_{(18)}\, k_0 - C_{(20)}\, h_0\} = \lambda\, C_{(19)}$

$\{C_{(20)/0} + (C_{(10)} - C_{(13)})\, k_0 - (C_{(9)} - C_{(16)})s_0 - C_{(17)}\, r_0$

$-(C_{(12)} - C_{(15)})\, t_0 - C_{(18)}\, u_0 + C_{(19)}\, h_0\} = \lambda C_{(20)}$ (3.5)

Multiplying equation (3.5) by g^{jk}, we can obtain on rearrangement

$P_i = L\, [\, m_i\{C_{(1)/0} + C_{(8)/0} + C_{(11)/0} + C_{(14)/0} + u_0(C_{(3)} + C_{(6)} + C_{(9)} + C_{(16)})$

$+ r_0(C_{(4)} + C_{(7)} + C_{(10)} + C_{(13)}) - h_0(C_{(2)} + C_{(5)} + C_{(12)} + C_{(15)})\} + n_i\{C_{(2)/0}$

$+ C_{(5)/0} + C_{(12)/0} + C_{(15)/0} + h_0(C_{(1)} + C_{(8)} + C_{(11)} + C_{(14)}) + s_0(C_{(4)} + C_{(7)}$

$+ C_{(10)} + C_{(13)}) - k_0(C_{(3)} + C_{(6)} + C_{(9)} + C_{(16)})\} + p_i\{C_{(3)/0} + C_{(6)/0} + C_{(9)/0}$

$+ C_{(16)/0} + k_0(C_{(2)} + C_{(5)} + C_{(12)} + C_{(15)}) + t_0(C_{(4)} + C_{(7)} + C_{(10)} + C_{(13)})$

$-u_0(C_{(1)} + C_{(8)} + C_{(11)} + C_{(14)})\} + q_i\{C_{(4)/0} + C_{(7)/0} + C_{(10)/0} + C_{(13)/0}$

$-r_0(C_{(1)} + C_{(8)} + C_{(11)} + C_{(14)}) - s_0(C_{(2)} + C_{(5)} + C_{(12)} + C_{(15)}) - t_0(C_{(3)}$

$+ C_{(6)} + C_{(9)} + C_{(16)})\}]$ (3.6)

which implies by virtue of (2.6)

$P_i = L\, (C_{/0}\, m_i + C\, h_0 - C\, u_0 - C\, r_0)$ (3.7)

Thus we can observe that for a P*-Finsler space we should have

$C\, h_0 = 0$, $C\, u_0 = 0$ and $C\, r_0 = 0$. As $C \neq 0$, we shall take $h_0 = 0$, $u_0 = 0$ and $r_0 = 0$. Hence we have

Theorem 3.1. **In a five-dimensional P*-Finsler space scalars $h_0 = u_0 = r_0 = 0$.**

Under above conditions, we can observe that P_{ijk} can be expressed as

$P_{ijk} = L[C_{(1)/0} m_i\, m_j\, m_k + \{C_{(2)/0} + 3(-C_{(9)}\, k_0 + C_{(10)}\, s_0)\}n_i\, n_j\, n_k + \{C_{(3)/0} + 3(C_{(12)}\, k_0$

$+ C_{(13)}\, t_0)\}p_i\, p_j\, p_k + \{C_{(4)/0} - 3(\, C_{(15)}\, s_0 + C_{(16)}\, t_0)\}q_i\, q_j\, q_k + \sum_{(i,j,k)} [m_i\, m_j\, n_k\, \{C_{(5)/0}$

$- C_{(6)}\, k_0 + C_{(7)}\, s_0\} + m_i\, m_j\, p_k\{C_{(6)/0} + C_{(5)}\, k_0 + C_{(7)}\, t_0\, \} + m_i\, m_j\, q_k\, \{C_{(7)/0}$

$- C_{(5)} s_0 - C_{(6)} t_0\} + n_i n_j m_k \{C_{(8)/0} - (C_{(2)} - 2 C_{(5)})h_0 - 2 C_{(17)} k_0 + 2 C_{(18)} s_0\}$

$+ n_i n_j p_k \{C_{(9)/0} + (C_{(2)} - 2C_{(12)} k_0 + C_{(10)} t_0 + 2 C_{(20)} s_0\} + n_i n_j q_k \{C_{(10)/0}$

$- (C_{(2)} - 2 C_{(15)})s_0 - C_{(9)} t_0 - 2 C_{(20)} k_0\} + p_i p_j m_k \{C_{(11)/0} + 2C_{(17)} k_0 + 2C_{(19)} t_0\}$

$+ p_i p_j n_k \{C_{(12)/0} -(C_{(3)} - 2 C_{(9)})k_0 + C_{(13)} s_0 + 2 C_{(20)} t_0\} + p_i p_j q_k \{C_{(13)/0}$

$-(C_{(3)} - 2 C_{(16)}) t_0 - C_{(12)} s_0 + 2 C_{(20)} k_0\} + q_i q_j m_k \{C_{(14)/0} - 2 C_{(18)} s_0$

$-2 C_{(19)} t_0\} + q_i q_j n_k \{C_{(15)/0} + (C_{(4)} - 2 C_{(10)}) s_0 - C_{(16)} k_0 - 2 C_{(20)} t_0\}$

$+ q_i q_j p_k \{C_{(16)/0} + (C_{(4)} - 2 C_{(13)})t_0 + C_{(15)} k_0 - 2 C_{(20)} s_0\}$

$+ (m_i n_j + m_j n_i) p_k\{C_{(17)/0} + (C_{(8)} - C_{(11)})k_0 + C_{(18)} t_0 + C_{(19)} s_0 \}$

$+ (m_i n_j + m_j n_i)q_k \{C_{(18)/0} - (C_{(8)} - C_{(14)}) s_0 - C_{(17)} t_0 - C_{(19)} k_0 + C_{(20)} u_0\}$

$+ (m_i p_j + m_j p_i)q_k \{C_{(19)/0} - C_{(17)} s_0 - (C_{(11)} - C_{(14)})t_0 + C_{(18)} k_0 \}$

$+ (n_i p_j + n_j p_i) q_k \{C_{(20)/0} + (C_{(10)} - C_{(13)}) k_0 - (C_{(9)} - C_{(16)})s_0 -(C_{(12)} - C_{(15)}) t_0\}]]$

C-Reducible and P-Reducible Finsler Spaces F^5

C-reducible Finsler spaces have been defined and studied by Matsumoto [5] and others. According to Matsumoto [5], a five-dimensional Finsler space F^5, will be C-reducible if it satisfies

$$C_{ijk} = (1/6) (h_{ij} C_k + h_{jk} C_i + h_{ki} C_j) \tag{4.1}$$

Comparing equations (2.5) and (4.1), we can obtain

$$C_{(1)} = LC/2,\ C_{(2)} = C_{(3)} = C_{(4)} = C_{(5)} = C_{(6)} = C_{(7)} = C_{(9)} = C_{(10)} = C_{(12)} = C_{(13)}$$

$$= C_{(15)} = C_{(16)} = C_{(17)} = C_{(18)} = C_{(19)} = C_{(20)} = 0,$$

$$C_{(8)} = C_{(11)} = C_{(14)} = LC/6. \tag{4.2}$$

Alternatively, equation (4.2) can also be expressed as [11]:

$$C_{222} = LC/2,\ C_{233} = C_{244} = C_{255} = LC/6,\ C_{223} = C_{224} = C_{225} = C_{234} = C_{235}$$

$$= C_{245} = C_{333} = C_{334} = C_{335} = C_{344} = C_{345} = C_{355} = C_{444} = C_{445} = C_{455} = C_{555} = 0.$$

Hence we have the usual result

Theorem 4.1. **A five-dimensional Finsler space F^5, will be C-reducible if and only if coefficients $C_{(1)}$ to $C_{(20)}$ satisfy equation (4.2).**

Following Matsumoto and Shimada [7] and Rastogi and Kawaguchi [14], a five- dimensional Finsler space will be P-reducible if P_{ijk} can be expressed as

$$P_{ijk} = (!/6)(A_{k/0} h_{ij} + A_{i/0} h_{jk} + A_{j/0} h_{ki}) \tag{4.3}$$

Comparing values of P_{ijk} in equations (3.3) and (4.3), we can obtain following non-zero terms

$$C_{(1)/0} + 3(C_{(6)} u_0 - C_{(5)} h_0 + C_{(7)} r_0) = (1/2) C_{/0},$$

$C_{(8)/0} – (C_{(2)} – 2 C_{(5)})h_0 + C_{(9)} u_0 + C_{(10)} r_0 – 2 C_{(17)} k_0 + 2 C_{(18)} s_0 = (1/6) C_{/0}$

$C_{(11)/0} + (C_{(3)} - 2 C_{(6)})u_0 - C_{(12)} h_0 + C_{(13)} r_0 + 2C_{(17)} k_0 + 2C_{(19)} t_0 = (1/6) C_{/0}$

$C_{(14)/0} + (C_{(4)} – 2 C_{(7)})r_0 – C_{(15)} h_0 + C_{(16)} u_0 – 2 C_{(18)} s_0 -2 C_{(19)} t_0 = (1/6) C_{/0}$

while all other terms in equation (3.3) shall vanish. Adding these equations and using equation (2.6) we get on simplification $h_0(C_{(5)} – C_{(15)}) = 0$. In case either $h_0 = 0$ or $C_{(5)} = C_{(15)}$, we shall get

$$C_{(1)/0} + C_{(8)/0} + C_{(11)/0} + C_{(14)/0} = C_{/0} \tag{4.4}$$

Hence we have

Theorem 4.2. **If a five -dimensional Finsler space is P-reducible, it satisfies equation (4.4).**

If the given five-dimensional P-reducible Finsler space is also P*-Finsler space, by virtue of $h_0 = r_0 = u_0 =0$, we shall get

$$C_{(1)/0} = C_{(8)/0}+ C_{(11)/0} + C_{(14)/0} = (1/2) C_{/0} \tag{4.5}$$

Hence we have

Theorem 4.3. **If a given five -dimensional Finsler space is both P-reducible as well as P*-Finsler space, it satisfies equation (4.5).**

V-Curvature Tensor in F^5

Corresponding to proposition (29.2) of Matsumoto [6] for F^3 and proposition in section 5 of Pandey and Dwivedi [10] for F^4, we give here following:

Proposition 5.1. Let T_{ij} be a skew-symmetric tensor of a five-dimensional Finsler space F^5, then for tensor

i) $*T^{ijk} = \varepsilon^{ijklm} T_{lm}/2$, we shall have $T_{ij} = \varepsilon_{ijklm} *T^{klm}$,

ii) For $T_{i0} = T_{ij} l^j = 0$, there will exist scalars α, β, γ, δ, Θ, φ, such that

$$T_{ij} = \alpha(m_i n_j – m_j n_i) + \beta(n_i p_j – n_j p_i) + \gamma(p_i q_j – p_j q_i)$$
$$+ \delta(q_i m_j – q_j m) + \Theta(m_i p_j–p_j m_i) + \varphi(n_i q_j – q_i n_j) \tag{5.1}$$

Proof. It is known that $\varepsilon_{\alpha\beta\gamma\delta\Theta}$ are scalar components of ε_{ijklm}, therefore i) is obvious. The condition $T_{i0} = 0$, is equivalent to $T_{\alpha 1} = 0$. The surviving scalar component of $T_{\alpha\beta}$ are $T_{23} = - T_{32}$, $T_{24} = - T_{42}$, $T_{34} = - T_{43}$, $T_{45} = - T_{54}$,$T_{52} = - T_{25}$, $T_{53} = - T_{35}$. Thus putting $\alpha = 2 T_{23}$, $\beta =2 T_{34}$, $\gamma = 2 T_{45}$, $\delta = 2 T_{52}$ and $\Theta = 2 T_{24}$, $\varphi = 2 T_{35}$, the proof of ii) will be completed.

To obtain V-curvature tensor of F^5, we observe that S_{hijk} is skew-symmetric in h, i as well as in j, k and $S_{0ijk} = S_{hi0k} = 0$, therefore by proposition (5.1), we get

$$L^2 S_{hijk} = [\alpha(m_h n_i – n_h m_i) + \beta(n_h p_i – p_h n_i) + \gamma(p_h q_i – q_h p_i)$$
$$+ \delta(q_h m_i – m_h q_i) + \Theta(m_h p_i – m_i p_h) + \varphi(n_h q_i – q_h n_i)].$$
$$[\alpha^{`} (m_j n_k – n_j m_k) + \beta^{`} (n_j p_k – p_j n_k) + \gamma^{`} (p_j q_k – q_j p_k)$$
$$+ \delta^{`} (q_j m_k – m_j q_k) + \Theta^{`}(m_j p_k – p_j m_k) + \varphi^{`}(n_j q_k – q_j n_k)], \tag{5.2}$$

where α, β, γ, δ, Θ, φ and α`,β`, γ`, δ`, Θ`, φ` are (0)-p-homogeneous scalars.

Since

$$S_{hijk} = C^r_{hk} C_{ijr} - C^r_{hj} C_{ikr} \tag{5.3}$$

therefore the scalar components of $S_{\alpha\beta\gamma\delta}$ of $L^2 S_{hijk}$ are also written as

$$L^2 S_{hijk} = S_{\alpha\beta\gamma\delta} e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k}, \tag{5.4}$$

where the tensor $S_{\alpha\beta\gamma\delta}$ in terms of $C_{\alpha\beta\gamma}$ can also be expressed as

$$S_{\alpha\beta\gamma\delta} = C_{\alpha\delta\Theta} C_{\Theta\beta\gamma} - C_{\alpha\gamma\Theta} C_{\Theta\beta\delta} \tag{5.5}$$

Due to skew-symmetry the surviving independent components of $S_{\alpha\beta\gamma\delta}$

Of five -dimensional Finsler space F^5, shall be given by following 21 components:

$S_{2323} = C_{23\Theta} C_{\Theta 32} - C_{22\Theta} C_{\Theta 33}$, $S_{2424} = C_{24\Theta} C_{\Theta 42} - C_{22\Theta} C_{\Theta 44}$,

$S_{2525} = C_{25\Theta} C_{\Theta 52} - C_{22\Theta} C_{\Theta 55}$, $S_{3434} = C_{34\Theta} C_{\Theta 43} - C_{33\Theta} C_{\Theta 44}$,

$S_{3535} = C_{35\Theta} C_{\Theta 53} - C_{33\Theta} C_{\Theta 55}$, $S_{4545} = C_{45\Theta} C_{\Theta 54} - C_{44\Theta} C_{\Theta 55}$,

$S_{2334} = C_{24\Theta} C_{\Theta 33} - C_{23\Theta} C_{\Theta 34}$, $S_{2345} = C_{25\Theta} C_{\Theta 34} - C_{24\Theta} C_{\Theta 35}$,

$S_{2324} = C_{24\Theta} C_{\Theta 32} - C_{22\Theta} C_{\Theta 34}$, $S_{2325} = C_{25\Theta} C_{\Theta 32} - C_{22\Theta} C_{\Theta 35}$,

$S_{2335} = C_{25\Theta} C_{\Theta 33} - C_{23\Theta} C_{\Theta 35}$, $S_{3424} = C_{34\Theta} C_{\Theta 42} - C_{32\Theta} C_{\Theta 44}$,

$S_{3425} = C_{35\Theta} C_{\Theta 42} - C_{32\Theta} C_{\Theta 45}$, $S_{3435} = C_{35\Theta} C_{\Theta 43} - C_{33\Theta} C_{\Theta 45}$,

$S_{3445} = C_{35\Theta} C_{\Theta 44} - C_{34\Theta} C_{\Theta 45}$, $S_{4524} = C_{44\Theta} C_{\Theta 52} - C_{42\Theta} C_{\Theta 54}$,

$S_{4525} = C_{45\Theta} C_{\Theta 52} - C_{42\Theta} C_{\Theta 55}$, $S_{4535} = C_{45\Theta} C_{\Theta 53} - C_{43\Theta} C_{\Theta 55}$,

$S_{5224} = C_{54\Theta} C_{\Theta 22} - C_{52\Theta} C_{\Theta 24}$, $S_{5235} = C_{55\Theta} C_{\Theta 23} - C_{53\Theta} C_{\Theta 25}$,

$S_{2435} = C_{25\Theta} C_{\Theta 43} - C_{23\Theta} C_{\Theta 45}$. (5.6)

Rearranging terms of equation (5.2), we get on simplification that the tensor S_{hijk} of a five-dimensional Finsler space F^5 can be expressed as Matsumoto [6]:

$$L^2 S_{hijk} = (h_{hj} M_{ik} + h_{ik} M_{hj} - h_{hk} M_{ij} - h_{ij} M_{hk}) \tag{5.7}$$,

where M_{ij} is an indicatory tensor.

CONCLUSIONS

We have discussed the properties of Torsion Tensor C ijk and P ijk & also the third curvature Tensor in five dimensional Finsler space .Further obtained very useful results, theorem 3.1, theorem 4.1, 4.2, 4.3 & proposition 5.1 .Other curvature tensors & their properties may be further obtained.

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